Metoda porównywania parami

Podejście HRE

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Advances in the Pairwise Comparisons Method

Heuristic Rating Estimation Approach

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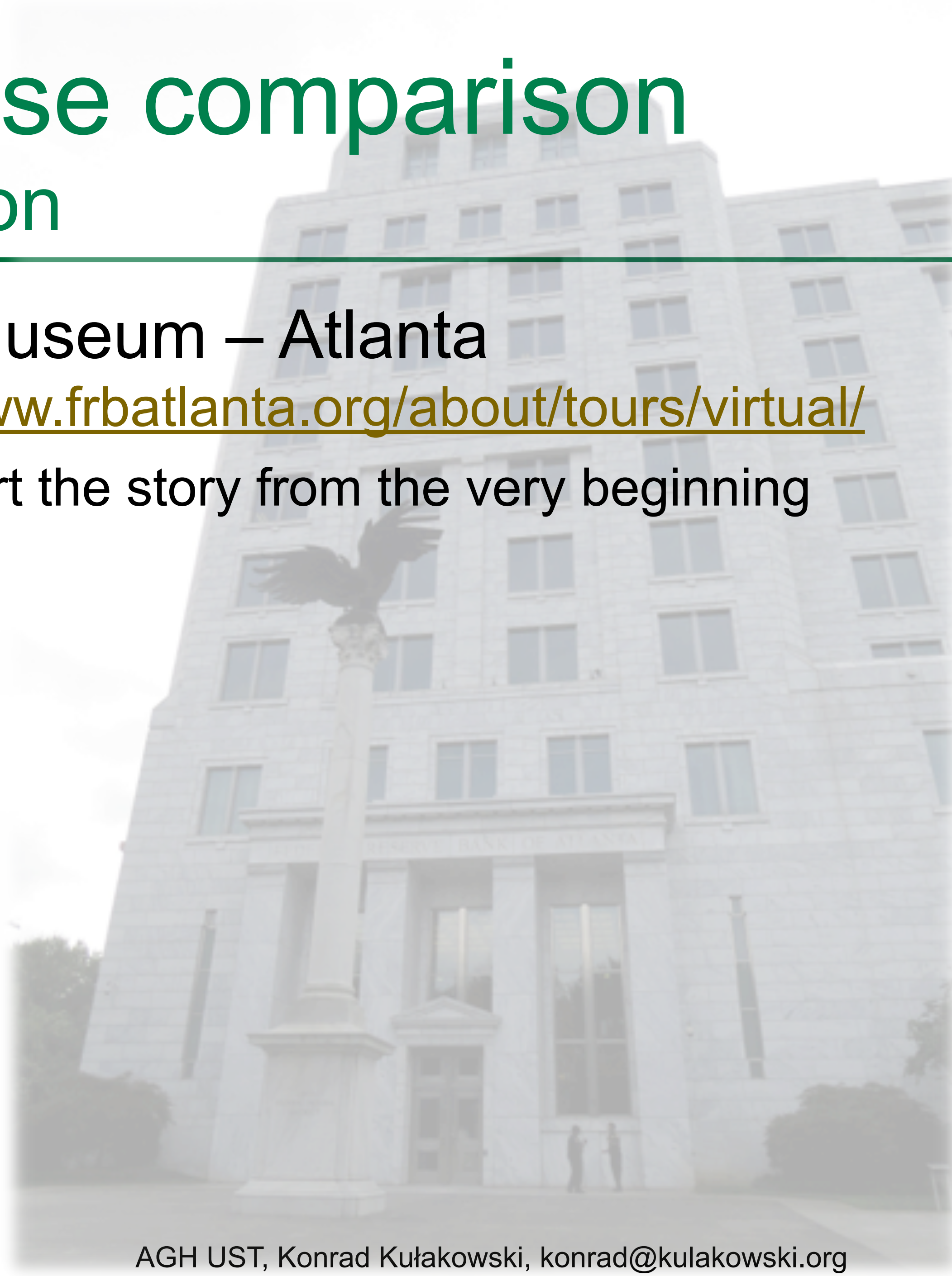
Outline

- Pairwise comparisons – motivation
- Classical approach
- Heuristic Rating Estimation (HRE) approach – motivation
- HRE method
- Possible areas of application
- Inconsistency and existence of solution in HRE
- Add-ons
- Bibliography

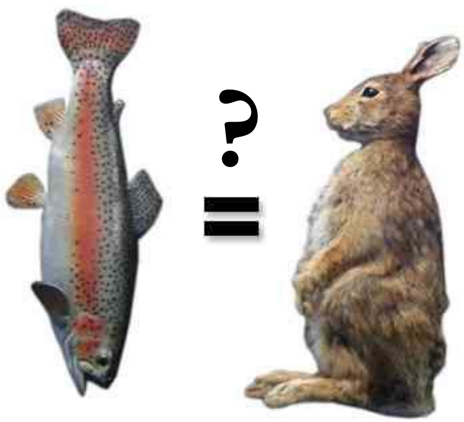
Pairwise comparison
motivation



- FED Museum – Atlanta
  http://www.frbatlanta.org/about/tours/virtual/
  - To start the story from the very beginning

Pairwise comparison
motivation

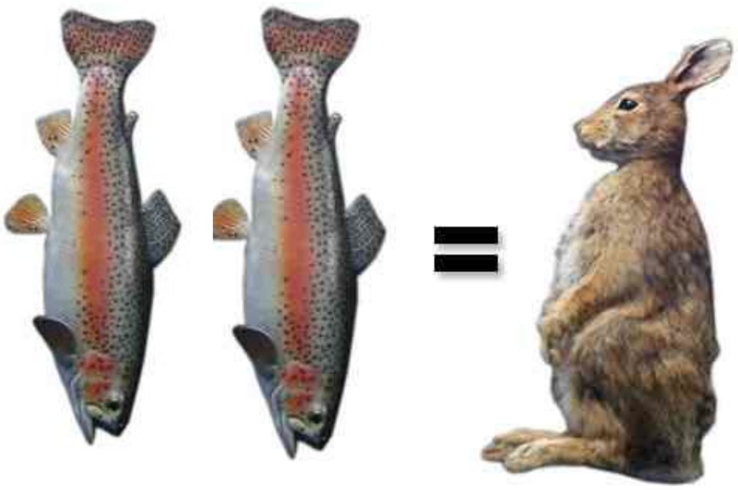
- Before money - barter
  (see FED Museum: http://www.frbatlanta.org/about/tours/virtual/money/)



- History of trade is in fact history of pairwise comparisons

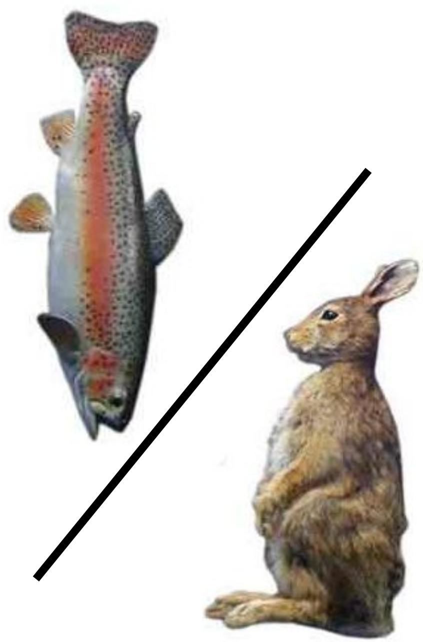
Pairwise comparison
motivation

- Barter - comparing incomparable
  - it's hard to judge the actual value of things, hence the judgment is always subjective



Pairwise comparison
motivation

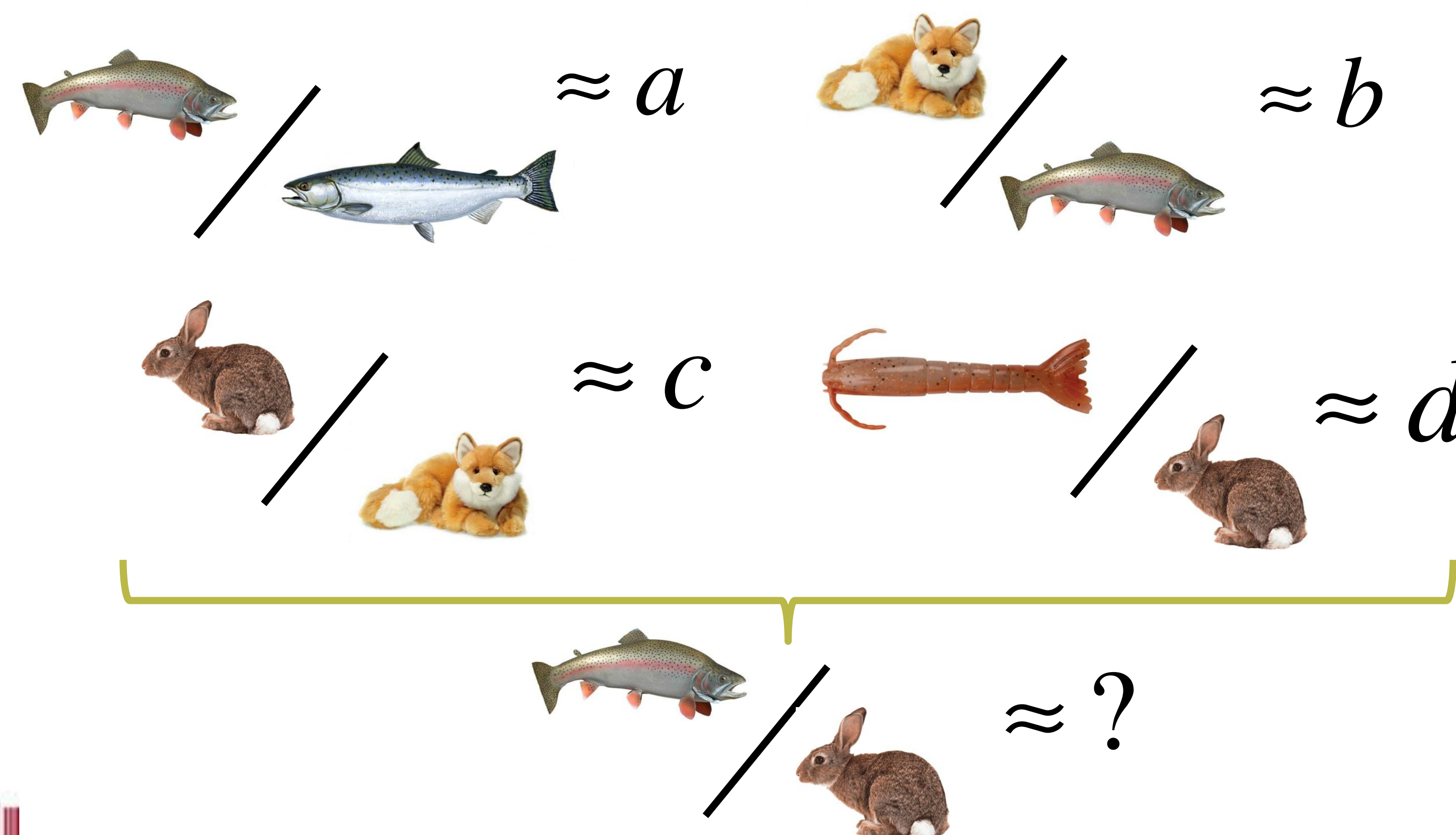
- Experts judgment implies relative value of goods:



$$\approx \frac{1}{2}$$

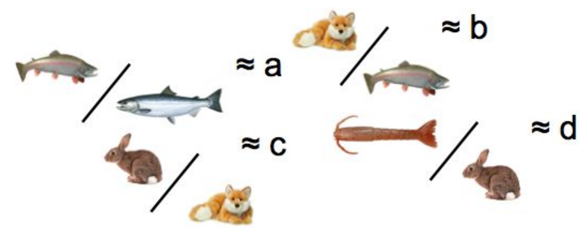
Pairwise comparison
motivation

- More comparisons

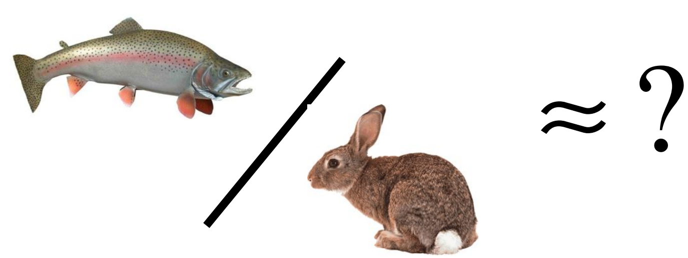


Pairwise comparison

motivation

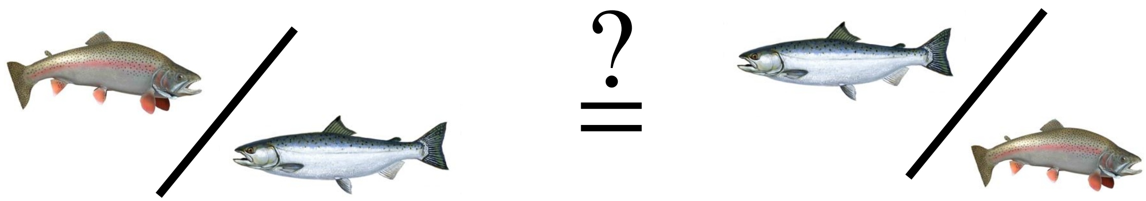


The need for methods of synthesis of partial results

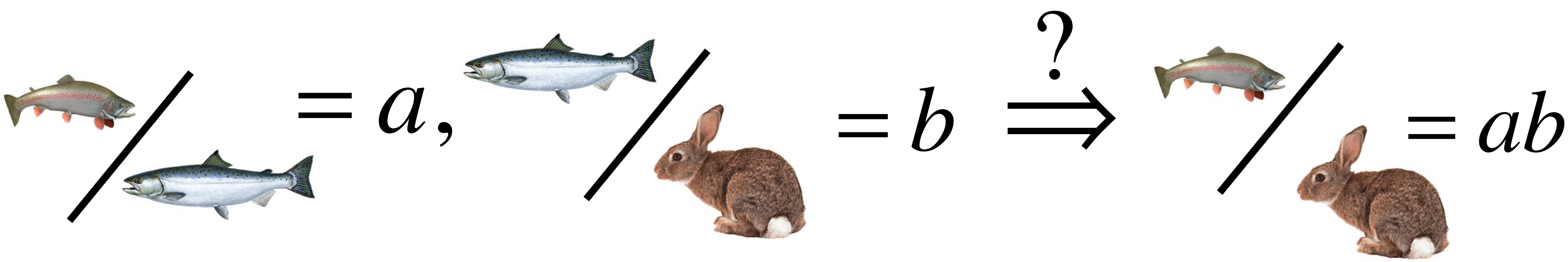


Pairwise comparison
motivation

- Problems with expert judgments
  - Reciprocity



  - Consistency



Pairwise comparison

Method

Pairwise comparisons (PC) method
Result synthesis

- Pairwise comparisons matrix

$$M = \begin{bmatrix} 1 & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & 1 & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & 1 & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & 1 & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & 1 \end{bmatrix}$$

where $m_{ij} \in \mathbb{R}_+$

PC method

Classical (eigenvalue based) approach

- Classical approach by Thomas Saaty
  - Assumption – matrix must be reciprocal i.e.

$$m_{ij} = \frac{1}{m_{ji}}$$

  - Value quantification:

$$m_{ij} \in \left\{ {}^{a}\!/\!{}_{b} : a, b \in \{1, \ldots, 10\} \right\}$$

  - Method of synthesis:
    - Finding a principal eigenvector x:

$$Mx = \lambda_{\max} x$$

Saaty, T. L., "A scaling method for priorities in hierarchical structures", Journal of Mathematical Psychology (1977), 234 - 281.

PC method
Classical approach

- Relative value of ,  ,  ,  ,  , 
  is given as:

$$x = [x_1, x_2, \ldots, x_5]$$

or even better, as rescaled principal eigenvector:

$$\hat{x} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_5]$$

where:

$$\hat{x}_i = x_i \Big/ \sum_{j=1}^{5} x_j$$

PC method
Classical approach

It is possible to determine the relative value of goods?

PC Method
Problem

- What happen when some goods are exchanged for money? E.g.:

=

=

PC Method
Problem

- What happen when some goods are exchanged for money? E.g.:

$= \$10$

$= \$20$

Then:

/

$= 1/2$

PC method

Standard approach

- Standard answer
  - Let us adopt that:

 /  $= \frac{1}{2}$

  - then calculate the numerical ranking of goods
- In result, every good will gain some weight (rank position)

PC method

Heuristic Rating Estimation (HRE) approach

- Let us do not (re) introduce the weights where they are given… (here the order is introduced by value in Dollars)

  but instead …

- Adopt the given values as reference and estimate the others

Kułakowski, K., "Heuristic Rating Estimation Approach to The Pairwise Comparisons Method", CoRR 2013, http://arxiv.org/abs/1309.0386 (to be appeared in Fundamenta Informaticae)

PC method

HRE approach

- Set of concepts:

$$C = C_K \cup C_U$$

- where
  - C_K means the set of concepts (alternatives) for which the value μ is initially known
  - C_U means the set of concepts (alternatives) for which μ need to be determined

PC method

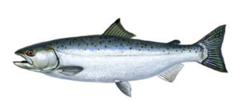
HRE approach

- HRE Input
  - Known concepts

$C_K = \{$  ,  $\}$ where

$\mu($  $) = 10,$ $\mu($  $) = 20$

  - Unknown concepts

$C_U = \{$  ,  ,  $\}$

PC method

HRE approach – result synthesis

- Pairwise comparisons matrix

$$
M = \begin{array}{c c}
 & \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \\
\begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{array} &
\begin{bmatrix}
1 & m_{12} & m_{13} & m_{14} & m_{15} \\
m_{21} & 1 & m_{23} & m_{24} & m_{25} \\
m_{31} & m_{32} & 1 & 2 & m_{35} \\
m_{41} & m_{42} & 1/2 & 1 & m_{45} \\
m_{51} & m_{52} & m_{53} & m_{54} & 1
\end{bmatrix}
\end{array}
$$

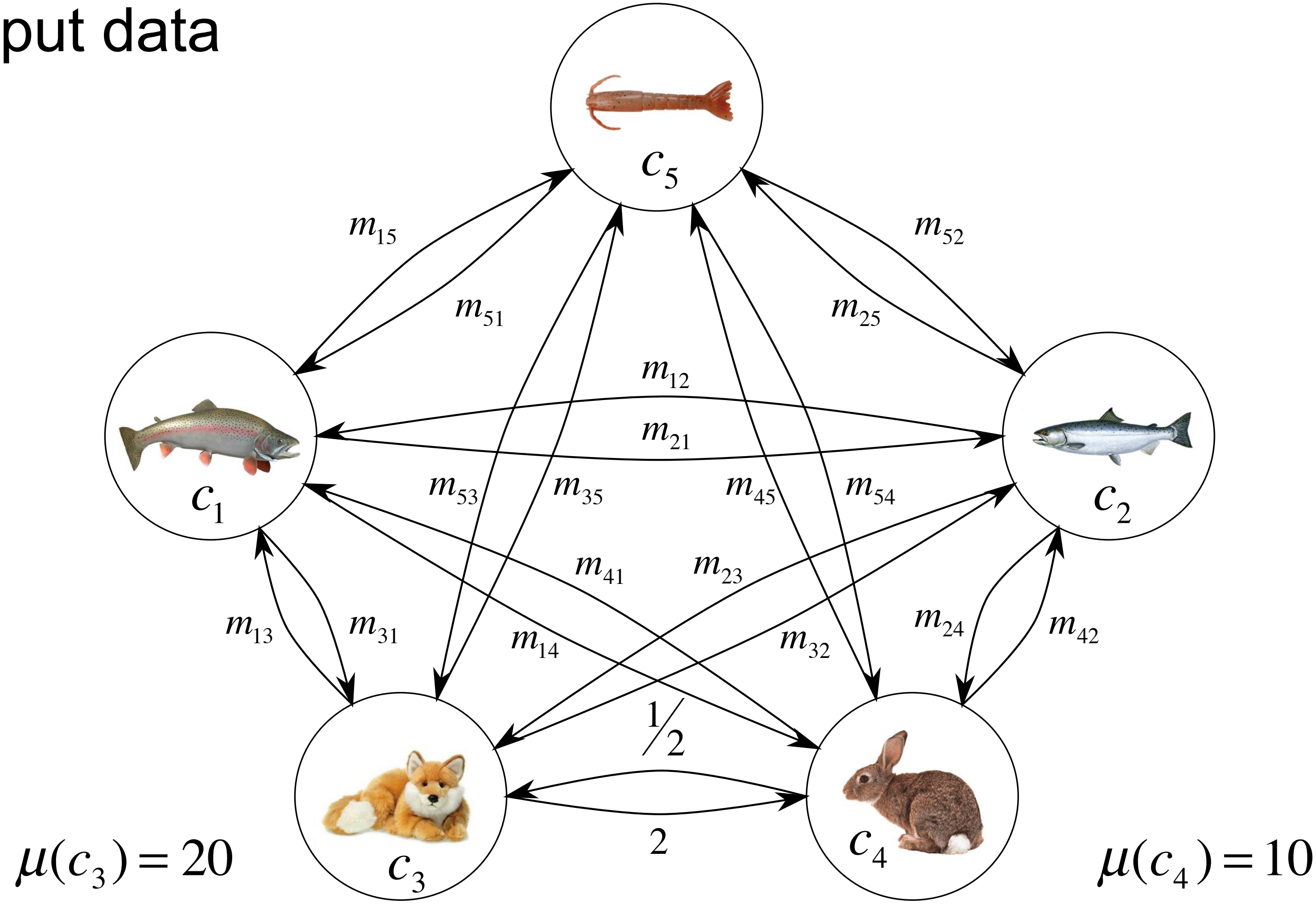
Known

where $m_{ij} \in \mathbb{R}_+$

PC method

HRE approach – result synthesis algorithm

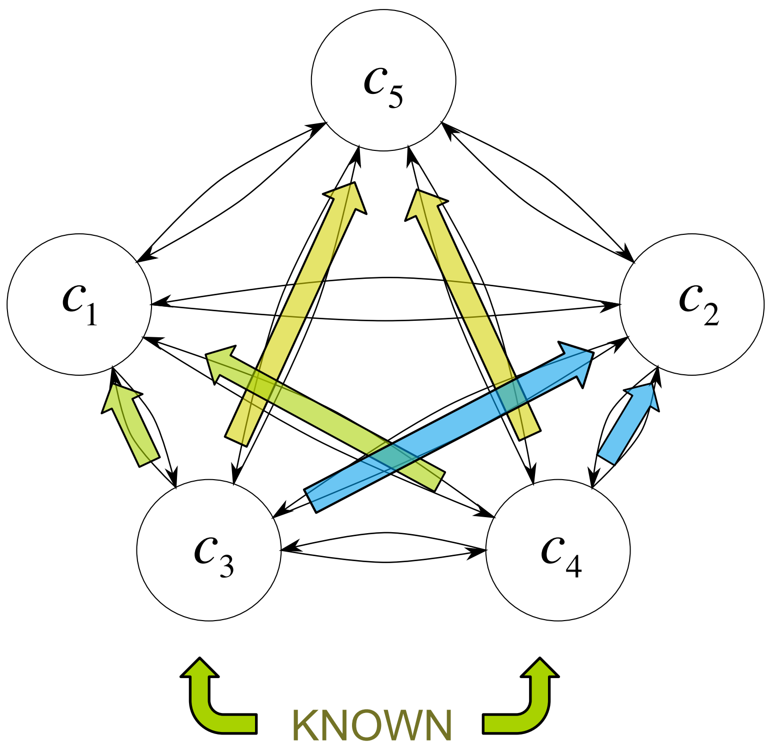
- Input data



PC method

HRE iterative approach – result synthesis algorithm

- First step



We may expect that:

$$\mu(c_1) = \frac{1}{2}\left(m_{13}\mu(c_3) + m_{14}\mu(c_4)\right)$$

$$\mu(c_2) = \frac{1}{2}\left(m_{23}\mu(c_3) + m_{24}\mu(c_4)\right)$$

$$\mu(c_5) = \frac{1}{2}\left(m_{53}\mu(c_3) + m_{54}\mu(c_4)\right)$$

PC method

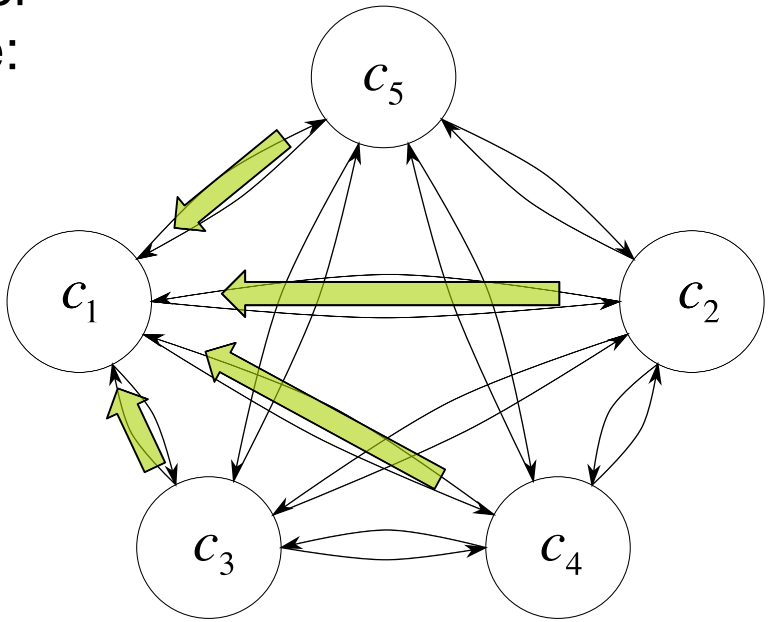
HRE iterative approach – result synthesis algorithm

- Second step and further
  - Let us take into account all other concepts but the estimated one:

$$\mu(c_1) = \frac{1}{4}\sum_{i=2}^{5} m_{1i}\mu(c_i)$$

  - and in general:

$$\mu(c_j) = \frac{1}{4}\sum_{i\in\{1,\ldots,5\}\setminus\{j\}} m_{ji}\mu(c_i)$$



PC method

HRE iterative approach – result synthesis algorithm

- Update equation
  - in $r = 0,1,2,\ldots$ update step

$$\mu_r(c_j) = \frac{1}{\left|C_j^{r-1}\right|} \sum_{c_i \in C_j^{r-1}} m_{ji}\mu_{r-1}(c_i)$$

  - where

$$C_j^{r-1} = \{c \in C : \mu_{r-1}(c) \text{ is defined and } c \neq c_j\}, \text{ and } C_j^0 \overset{df}{=} C_K$$

PC method

HRE iterative approach – result synthesis algorithm

- Assuming that $C_K = \{c_{k+1}, \ldots, c_n\}$
- … the relative value of concepts is given as:

$$\mu = [\underbrace{\mu(c_1), \ldots, \mu(c_k)}_{C_U}, \underbrace{\mu(c_{k+1}), \ldots, \mu(c_n)}_{C_K}]$$

(HRE procedure) (a priori known)

- Rescaling (normalization)

$$\hat{\mu} = \left[\hat{\mu}(c_1), \ldots, \hat{\mu}(c_n)\right]$$

where $$\hat{\mu}(c_i) = \mu(c_i) \Big/ \sum_{j=1}^{n} \mu(c_j)$$

PC method
HRE iterative approach

What is μ when the number of iterations tends to ∞?

PC method

From iterative procedure to linear equation

- The algorithm as shown above follows the Jacobi iterative method for the linear equation system in form:

$$A\mu = b$$

where:

- A is a matrix of $r \times r$ where $r = |C| - |C_K| = |C_U|$
- b is a vector of constant terms
- μ is a vector:

$$\mu^T = [\mu(c_1), \ldots, \mu(c_k)]$$

PC method

From iterative procedure to linear equation

- b - vector of constant terms is given as:

$$b = \begin{bmatrix} \frac{1}{n-1} m_{1,k+1}\mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{1,n}\mu(c_n) \\ \frac{1}{n-1} m_{2,k+1}\mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{2,n}\mu(c_n) \\ \vdots \\ \frac{1}{n-1} m_{k,k+1}\mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{k,n}\mu(c_n) \end{bmatrix}$$

  - note that (assumed for the sake of simplicity):

$$C = \{c_1, \ldots, c_n\} \text{ and } C_K = \{c_{k+1}, \ldots, c_n\}$$

PC method

From iterative procedure to linear equation

- The matrix A used by the HRE approach is:

$$A = \begin{bmatrix} 1 & -\frac{1}{n-1}m_{1,2} & \cdots & \cdots & -\frac{1}{n-1}m_{1,k} \\ -\frac{1}{n-1}m_{2,1} & 1 & -\frac{1}{n-1}m_{2,3} & & -\frac{1}{n-1}m_{2,k} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ -\frac{1}{n-1}m_{k,1} & \cdots & \cdots & -\frac{1}{n-1}m_{k,k-1} & 1 \end{bmatrix}$$

PC method
Direct solving

- The equation:

$$A\mu = b$$

can also be solved directly….

PC method

Direct solving – theorems and facts

- Theorem 1
  - Equation $A\mu = b$ has exactly one solution if $\det(A) \neq 0$
- Theorem 2
  - The Jacobi method is convergent if A is strictly diagonally dominant by rows i.e.

$$1 > \sum_{j=1, j\neq i}^{k} |a_{ij}|$$

for $i = 1, \ldots, k$

PC method

Direct solving – admissibility of solution

- Observation - result must be positive
  - i.e. $\mu(c_i) > 0$ for $i = 1, \ldots, n$
  - That is because of the form of update equation:

$$\mu_r(c_j) = \frac{1}{\left|C_j^{r-1}\right|} \sum_{c_i \in C_j^{r-1}} m_{ji} \mu_{r-1}(c_i)$$

- Observation - A is strictly diagonally dominant, then μ obtained using direct method is admissible

PC method

Direct solving – solution admissibility

- Is A diagonally dominant?
  - i.e.

$$1 > \sum_{j=1, j\neq i}^{k} \left| a_{ij} \right| \text{ where } a_{ij} = -\frac{1}{n-1} m_{ij}$$

- Observation
  - A has a high chance to be diagonally dominant when a_{ij} are not to large.
- Conclusion
  - The more concepts in C_K and the more similar to each other the better (the more likely A is diagonally dominant)

PC method

Intuitiveness of assumptions

- The more concepts in C_K
  - Corresponds to the natural desire to have more than the lower number of reference concepts
- The more similar to each other….
  - Humans (experts) are able to compare (judge) similar things but not different.
- Observation
  - In most cases tested the matrix A was diagonally dominant

PC method, HRE approach
Applications

How to?

PC method, HRE approach

How to?

- Define the problem
  - Specify problem domain, the set of concepts and the partial function μ
- Define the reference set C_K
  - assign the μ values to the concepts from C_K
- Gather the reference data
- Apply the HRE method
- To remember - this is only heuristics
  - above all, common sense!

PC method, HRE approach
Applications

Areas of application

Decision Support Systems

PC method, HRE approach
Possible areas of application

- An optimal drug (treatment) selection
  - Reference set C_K
    - Group of drugs with proven efficacy in clinical trials
  - Other concepts $C_U = C \setminus C_K$ and M
    - Comparative opinions of experts (physicians) based on their clinical experience

PC method, HRE approach

Possible areas of application

- Introducing new products into the market
  - Reference set C_K
    - existing products with the sales statistics
  - New product candidates $C_U = C \setminus C_K$
    - Comparative opinion survey of the target group

PC method, HRE approach

Possible areas of application

- Support for assessment of the value of companies
  - Joint stock companies – reference set
    - an actual value is determined by stock exchange
  - Companies outside the exchange trading – the rest of concepts
    - Comparative company value estimation

PC method, HRE approach

Possible areas of application

- Real Estate Valuation
  - Reference set C_K
    - real estates in the given area with the known transaction price
  - Real estates $C_U = C \setminus C_K$ for which the prices are unknown
    - Comparative (paired) judgments of appraiser

PC method, HRE approach

Possible areas of application

- Painting Valuation
  - Reference set C_K
    - paintings from the given period with the known transaction price
  - Paintings $C_U = C \setminus C_K$ for which the prices are unknown
    - Comparative (paired) judgments of appraiser

PC method, HRE approach
Possible areas of application

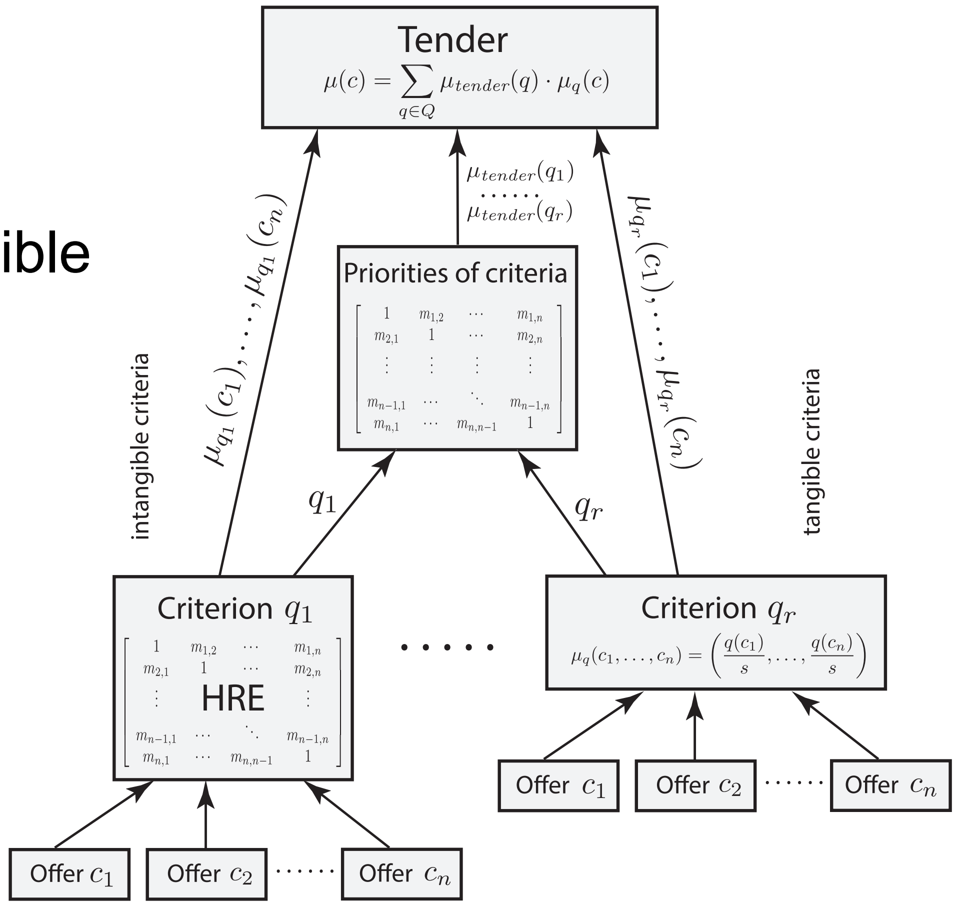
- Tender assessments
  - Reference set C_K
    - group of the winning bids from previous tenders, reference offers
    - investment estimates
  - Other concepts $C_U = C \setminus C_K$ and M
    - Comparative opinions of experts (offers evaluation committee) based on their knowledge and the lessons learned in the past

Kułakowski, K., J. Szybowski, R. Tadeusiewicz, "Tender with success the pairwise comparisons approach", KES 2014, (to be appeared in Procedia in Computer Science, Elsevier)

PC method, HRE approach

Possible areas of application

- Hierarchical tender assessment model
  - success criterion
  - tangible and intangible criteria



PC method, HRE approach

Possible areas of application

- anywhere the expert judgment is important
  - for example when accurate model would be very complex and difficult to construct
    - e.g. to estimate the rate of return on investment in green technologies

PC method, HRE approach

Possible areas of application

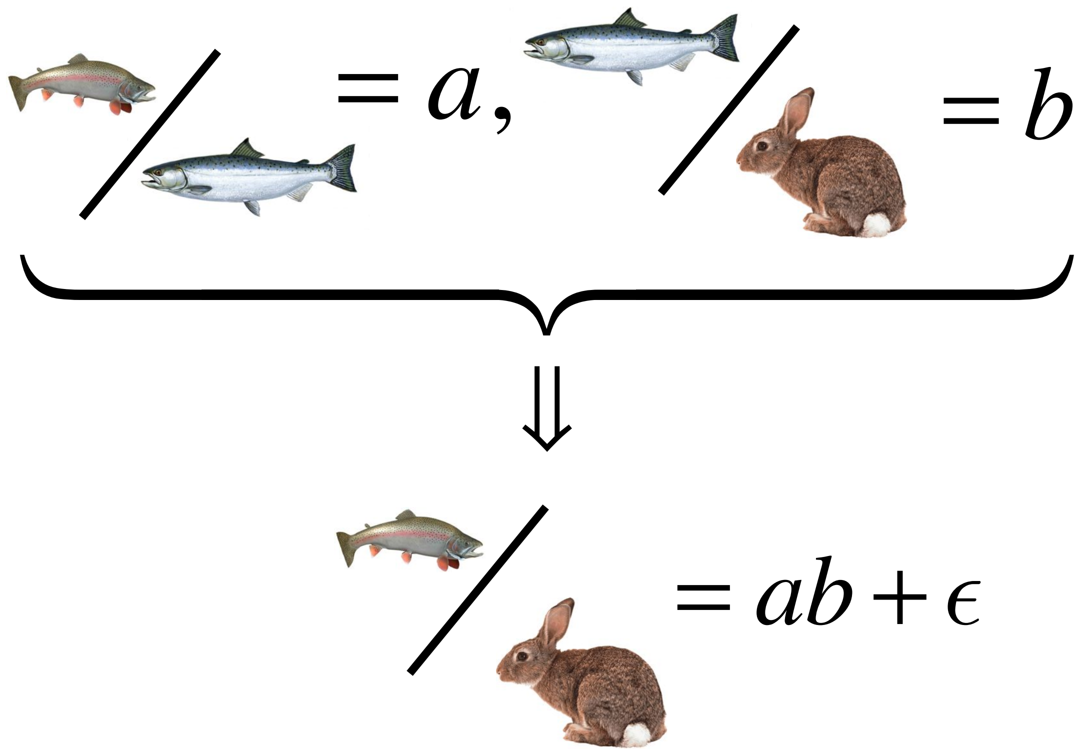
and many more

Inconsistency and HRE approach

Inconsistency, HRE approach
and existence of solution

Inconsistency and HRE approach

- What is inconsistency?



- The smaller ϵ the smaller inconsistency…

Inconsistency and HRE approach
Inconsistency indices

- Saaty's inconsistency index:

$$IC = \frac{\lambda_{max} - n}{n - 1}$$

where λ_{max} is a principal eigenvalue of some matrix

$$M = (m_{ij}) \wedge m_{ij} \in \mathbb{R}_+ \wedge i, j \in \{1, \ldots, n\}$$

Inconsistency and HRE approach
Inconsistency indices

- Koczkodaj's triad (local) inconsistency:

$$\kappa_{i,j,k} \overset{df}{=} \min\left\{\left|1-\frac{m_{ij}}{m_{ik}m_{kj}}\right|,\left|1-\frac{m_{ik}m_{kj}}{m_{ij}}\right|\right\}$$

- Koczkodaj's inconsistency index:

$$\mathcal{K}(M)=\max_{i,j,k\in\{1,\ldots,n\}}\left\{\kappa_{i,j,k}\right\}$$

Inconsistency and HRE approach
Existence of solution

- The linear equation system $A\mu = b$ introduced in the HRE approach has exactly one strictly positive solution for $0 < r \leq n-2$ if

$$\mathcal{K}(M) < 1 - \frac{1 + \sqrt{1 + 4(n-1)(n-r-2)}}{2(n-1)}$$

where $n = |C_U \cup C_K|$ is the number of all the estimated concepts, whilst $r = |C_K|$ is the number of the known concepts.

Kułakowski, K., "Notes on the existence of solution in the pairwise comparisons method using the Heuristic Rating Estimation approach" CoRR 2014, http://arxiv.org/abs/1402.4064

Inconsistency and HRE approach

Proof idea (1), M-matrices

- *M-matrix* definition
  - An $n \times n$ matrix that can be expressed in the form $A = sI - B$ where $B = [b_{ij}]$ with $b_{ij} \geq 0$ for $i, j \in \{1,\ldots,n\}$ and $s \geq \rho(B)$ is called *M-matrix*.
  - where:

$$\rho(B) = \max\{|\lambda| : \det(\lambda I - B) = 0\}$$

  - is a spectral radius of B (the maximum of the moduli of the eigenvalues of B

Inconsistency and HRE approach

Proof idea (2), M-matrices

- Plemmons' Theorem
  - For every $A = [a_{ij}]$ - $n \times n$ matrix over $\mathbb{R}$ where $a_{ij} \leq 0$ and $i \neq j, i, j = 1, \ldots, n$ each of the following conditions is equivalent to the statement: A is a nonsingular *M-matrix*.
    - A is inverse positive. That is A^{-1} exists, and $A^{-1} \geq 0$
    - A is semi-positive. That is, there exists $x \geq 0$ with $Ax > 0$
    - There exists a positive diagonal matrix D such that AD has all positive row sums

Plemmons, R. J., "M-matrix characterizations. I - nonsingular M-matrices", Linear Algebra and its Applications (1976), 175--188.

Inconsistency and HRE approach

Proof idea (3), existence of solution

- Let us denote:

$$A = \begin{bmatrix} t_{1,1} m_{1,k} m_{k,1} & \cdots & -\dfrac{m_{1,k}}{n-1} \\ \vdots & \vdots & \vdots \\ -\dfrac{t_{k-1,1} m_{k-1,k} m_{k,1}}{n-1} & \ddots & -\dfrac{m_{k-1,k}}{n-1} \\ -\dfrac{t_{k,1} m_{k,1}}{n-1} & \cdots & 1 \end{bmatrix}$$

where

$$1 - \mathcal{K}(M) \leq t_{ij} \leq \frac{1}{1 - \mathcal{K}(M)}$$

Inconsistency and HRE approach
Proof idea (4), Existence of solution

- Matrix A can be written as a product $A = BC$
  - where:

$$B = \begin{bmatrix} t_{1,1}m_{1,k} & \cdots & \cdots & -\frac{m_{1,k}}{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ -\frac{t_{k-1,1}m_{k-1,k}}{n-1} & \vdots & t_{k-1,k-1}m_{k-1,k} & -\frac{m_{k-1,k}}{n-1} \\ -\frac{t_{k,1}}{n-1} & \cdots & -\frac{t_{k,k-1}}{n-1} & 1 \end{bmatrix} \quad C = \begin{bmatrix} m_{k,1} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & m_{k,k-1} & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$$

  - Adopting $D = I$ and applying the third Plemmons criterion leads to the thesis of the Theorem.

Inconsistency and HRE approach
Existence of solution

- Values of $\mathcal{K}(M)$ that guarantees existence of solution in the HRE approach

$\mathcal{K}(M)$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
$n=3$	0.5	-	-	-	-
$n=4$	0.232	0.666	-	-	-
$n=5$	0.156	0.359	0.75	-	-
$n=6$	0.118	0.259	0.441	0.8	-
$n=7$	0.095	0.204	0.333	0.5	0.833

PC method, HRE approach

Exploration areas

Add-ons

PC method, HRE approach
Further research, opportunities

- Non-reciprocal PC matrix heuristics (proposal)
  - Lack of reciprocity:

$$m_{ij} \neq \frac{1}{m_{ji}}$$

  - Let us transform: $M \to \hat{M}$
    - where

$$\hat{m}_{ij} = \left(m_{ij} \frac{1}{m_{ji}} \right)^{1/2}$$

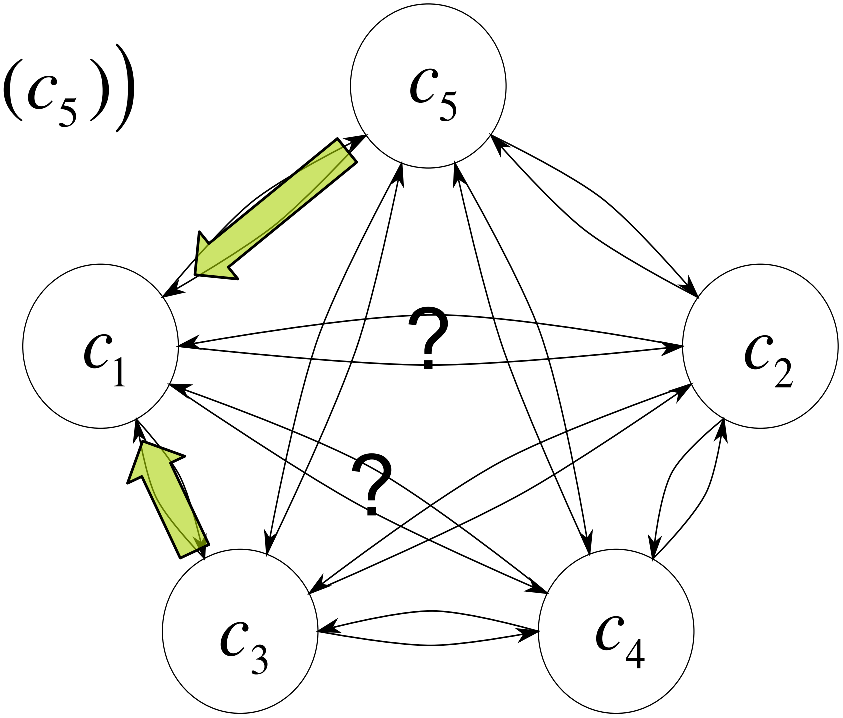
    - Observations:
      - $\hat{M}$ is reciprocal
      - if M is reciprocal then $M = \hat{M}$

PC method, HRE approach

Further research, opportunities

- Incomplete PC matrix heuristics (proposal)

$$\mu(c_1) = \frac{1}{2}\left(m_{13}\mu(c_3) + m_{15}\mu(c_5)\right)$$



- Impacts the lack of reciprocity heuristics

PC method, HRE approach

Further research, opportunities

- Minimizing estimation error heuristics
  - Average absolute estimation error

$$\hat{e}_{\mu} = \frac{1}{|C_U|} \sum_{c \in C_U} e_{\mu}(c)$$

  - Absolute estimation error

$$e_{\mu}(c_j) = \frac{1}{|C_j^{r-1}|} \sum_{c_i \in C_j^{r-1}} \left| \mu(c_j) - \mu(c_i) \cdot m_{ji} \right|$$

PC method, HRE approach

Wrapping up

Invitation

- KES Conference 2014
  - more at www.kulakowski.org/KES2014-PC

KES2014 IS13 Session

Modelling with Qualitative and Quantitative Pairwise Comparisons

Call for papers | KES Conference | Venue | Gdynia Maritime University

Pairwise comparisons

Weights or weighted parameters/attributes are part of most measurement, indexing and/or classification techniques. However when judgments are subjective, weight assignment and especially weights consistency is almost always problematic. A ranking or preference is usually defined as a weakly ordered relationship between a set of items such that, for any two items, the first is either "less preferred", "more preferred" or "indifferent" to the second one. Weights and ranking are dual concepts and usually one can be derived from another. Consistent subjective ranking is problematic and assigning numerical importance is even more problematic.

The pairwise comparisons method is based on the observation that it is much easier to judge the mutual relationship (preference, importance, intensity, etc.) of two objects than to do this for several objects. This very old idea goes back to Ramon Llull in the end of XIII century. It was made popular by an influential paper of Marquis de Condorcet (1785), where he used this method in the election process where voters rank candidates based on their preference. Modern version follows from Thurstone (1927) and Saaty (1977), and it is one of many methods used in multicriteria decision making and analysis.

While most existing methods involve numbers, in many cases using only qualitative assessment might be more trustworthy. Formulas and rules involving numbers are considered as more scientific and credible than those that involve qualitative values only. This is obviously true when the notions of interest can be measured directly or indirectly, as for instance velocity, height, voltage, pressure etc. However, when it comes to subjective notions as love, importance, taste, beauty, etc., we have to be very careful when numbers are used. Models that use concepts from abstract algebra, set theory, various types of logics are able successfully to model qualitative notions (and their calculi) as well.

PC method, HRE approach

Mathematica Package

- Pairwise Comparisons Mathematica Package powered by Raspberry PI
  - https://code.google.com/p/pairwise-comparisons/

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One more use of pairwise comparisons

(found on the web)

taken from http://sotak.info/sci.jpg

Questions